Gundy-Varopoulos martingale transforms and their projection operators

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Riesz transforms on $\mathbb{R}^d$

On $\mathbb{R}^d$, Riesz transforms R_j, $j = 1, \cdots, d$, are formally defined by

$$R_j = \partial_j (-\Delta)^{-1/2}.$$

The classical Calderón-Zygmund theory gives

$$\|R_j f\|_p \leq C_p \|f\|_p.$$

The sharp inequality can be obtained by either analytic or probabilistic approach.

Theorem (Bañuelos-Wang 1995, Iwaniec-Martin 1996)

$$\|R_j f\|_p \leq \cot\left(\frac{\pi}{2p^*}\right) \|f\|_p, \quad \forall p > 1$$

where $p^* = \max\{p, \frac{p}{p-1}\}$.

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In 1979, Gundy and Varopoulos proved the now classical representation of Riesz transforms using "background radiation" process:

$$R_j f = -2 \lim_{y_0 \to \infty} \mathbb{E}_{y_0} \left(\int_0^\tau A_j(\nabla, \partial_y)^{\mathrm{T}} Qf(\beta_s, B_s)(d\beta_s, dB_s) \mid \beta_\tau = x \right),$$

where

- $A_j = (a_{ik})$ is a $(d+1) \times (d+1)$ matrix with $a_{(d+1)j} = 1$ and otherwise 0;
- β_t: Brownian motion on $\mathbb{R}^d$ with initial distribution dx;
- B_t: Brownian motion on $\mathbb{R}$ with generator $\frac{d^2}{dy^2}$ starting from $y_0 > 0$;
- $\tau = \inf\{t > 0 : B_t = 0\}$: the stopping time;
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Sharp inequalities for martingales

Bañuelos and Wang proved the following sharp inequality extending the classical results of Burkholder (1966).

Theorem (Bañuelos-Wang 1995)

Let X and Y be two martingales with continuous paths such that Y is differentially subordinate to X. Fix $1 < p < \infty$, then

$$\|Y\|_p \leq (p^* - 1)\|X\|_p.$$

Furthermore, suppose the martingales X and Y are orthogonal. Then

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Scalar operators constructed from martingale transforms

$\mathbb{M}$: smooth manifold with smooth measure μ.

$X_1, \cdots, X_d$: locally Lipschitz vector fields on $\mathbb{M}$.

$V : \mathbb{M} \to \mathbb{R}$ is a non-positive smooth potential. Consider the Schrödinger operator

$$L = -\sum_{i=1}^{d} X_i^* X_i + V.$$

We can write

$$L = \sum_{i=1}^{d} X_i^2 + X_0 + V,$$

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Diffusion process

Let $(Y_t)_{t\geq 0}$ be the diffusion process on $\mathbb{M}$ with generator $\sum_{i=1}^{d} X_i^2 + X_0$ starting from the distribution μ.

Via Stratonovitch stochastic differential equation,

$$dY_t = X_0(t)dt + \sum_{i=1}^{d} X_i(Y_t) \circ d\beta_t^i,$$

where β_t is the Brownian motion on $\mathbb{R}^d$.

For $f \in C_0^\infty(\mathbb{M})$, denote

$$Q^V f(x,y) = P_y f(x) = e^{-y\sqrt{-L}} f(x).$$

Then $M_t^f = e^{\int_0^{t\wedge\tau} V(Y_s)ds} Q^V f(Y_{t\wedge\tau}, B_{t\wedge\tau})$ is a martingale.

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Projection operators

Consider the operators

$$T_i = \int_0^{+\infty} y P_y \left(\sqrt{-L} X_i - X_i^* \sqrt{-L} \right) P_y dy, \quad \forall 1 \le i \le d.$$

By Itô isometry and Itô's formula,

Theorem (Bañuelos-Baudoin-C. 2018)

For $f \in \mathcal{S}(\mathbb{M})$ *and* $1 \le i \le d$,

$$T_i f(x) = -\frac{1}{2} \lim_{y_0 \to \infty} \mathbb{E}_{y_0} \left(e^{\int_0^\tau V(Y_v) dv} \int_0^\tau e^{-\int_0^s V(Y_v) dv} A_i(\nabla, \partial_y)^{\mathrm{T}} Q f(Y_s, B_s)(d\beta_s, dB_s) \mid Y_\tau = x \right)$$

where $\nabla = (X_1, \cdots, X_d)$, *and* A_i *is a* $(d+1) \times (d+1)$ *matrix with* $a_{i(d+1)} = -1$, $a_{(d+1)i} = 1$ *and otherwise* 0.

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Sharp estimate of Bañuelos and Osękowski

Theorem (Bañuelos-Osękowski 2015)

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$$Z_t = e^{\int_0^t V_s ds} \int_0^t e^{-\int_0^s V_v dv} dY_s,$$

where $(V_t)_{t\geq 0}$ is a non-positive adapted and continuous process.

$$\|Z\|_p \leq (p^* - 1)\|X\|_p.$$

Main result

Theorem (Bañuelos-Baudoin-C. 2018)

Fix $1 < p < \infty$. *Then for every* $f \in \mathcal{S}(\mathbb{M})$,

$$\|T_i f\|_p \leq \left(\frac{3}{2}\right)(p^* - 1)\|f\|_p.$$

If the potential $V \equiv 0$, *then*

$$\|T_i f\|_p \leq \frac{1}{2}\cot\left(\frac{\pi}{2p^*}\right)\|f\|_p.$$

- **Applications:** Lie group of compact type, Heisenberg groups, $\mathbb{SU}(2)$, etc.
- **Generalizations:** T_A; Riesz transforms on vector bundles (forms, spinors).

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- **Generalizations:** T_A; Riesz transforms on vector bundles (forms, spinors).

Example: Lie group of compact type

G: Lie group of compact type with a bi-invariant Riemannian structure.

$X_1, \cdots, X_d$: an orthonormal basis of $\mathfrak{g}$.

$L = \sum_{i=1}^{d} X_i^2$: the Laplace-Beltrami operator.

We observe

$$T_i = \int_0^{+\infty} y P_y \left(\sqrt{-L} X_i - X_i^* \sqrt{-L} \right) P_y dy = \frac{1}{2} X_i (\sqrt{-L})^{-1}.$$

Proposition

$$\| X_i (\sqrt{-L})^{-1} \|_{L^p \to L^p} \leq \cot \left(\frac{\pi}{2p^*} \right).$$

This inequality was first proved by [Accozzi 1998].

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$$\mathbb{H}^n = \{(x, y, z) : x \in \mathbb{R}^n, y \in \mathbb{R}^n, z \in \mathbb{R}\}$$

endowed with the group law

$$(x, y, z)\cdot(x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}\left(\langle x, y'\rangle_{\mathbb{R}^n} - \langle y, x'\rangle_{\mathbb{R}^n}\right)\right).$$

Let

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_z, \quad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_z, \quad Z = \partial_z,$$

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$$[X_j, Y_k] = \delta_{jk} Z.$$

The complex gradient

$$W_j = X_j + iY_j.$$

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The sublaplacian

$$L = \sum_{j=1}^{n} \left(X_j^2 + Y_j^2\right).$$

By spectral decomposition of the sublaplacian,

$$[W_j, \sqrt{-L}]f = 2i\,\mathcal{T}_j Z f, \quad \forall f \in \mathcal{S}(\mathbb{H}^n)$$

where

$$\mathcal{T}_j = \int_0^{+\infty} y P_y (W_j \sqrt{-L} + \sqrt{-L} W_j) P_y dy.$$

Proposition

Let $1 \leq j \leq n$ *and* $f \in \mathcal{S}(\mathbb{H}^n)$. *Then we have*

$$\| \, [W_j, \sqrt{-L}]f \, \|_p \leq \sqrt{2}(p^* - 1) \|Zf\|_p.$$

The sublaplacian

$$L = \sum_{j=1}^{n} \left(X_j^2 + Y_j^2 \right).$$

By spectral decomposition of the sublaplacian,

$$[W_j, \sqrt{-L}]f = 2i\,\mathcal{T}_j Z f, \quad \forall f \in \mathcal{S}(\mathbb{H}^n)$$

where

$$\mathcal{T}_j = \int_0^{+\infty} y P_y (W_j \sqrt{-L} + \sqrt{-L} W_j) P_y dy.$$

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Thank you very much!