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A regularization property of heat semigroups and its applications

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Let (X, μ) be a measurable space (e.g. Riemannian manifolds, graphs or Dirichlet spaces) equipped with a self-adjoint operator L. Let $\{P_t\}_{t>0}$ be the associated heat semigroup. We are interested in the following regularization property of the heat semigroup: for $1 < p \leq \infty$,

$$(G_p) \qquad \| \, |\nabla P_t f| \, \|_p \leq \frac{C}{\sqrt{t}} \|f\|_p,$$

where ∇ denotes the gradient on Riemannian manifolds or its proper substitutes in other settings like "carré du champ".

To fix idea we consider complete Riemannian manifolds.

- (G_2) is always true by spectral theory.
- (G_∞) is the so-called weak Bakry-Émery curvature condition.
- When $2 < p \leq \infty$, (G_p) is linked to the geometry of the underlying space, Riesz transforms, harmonic functions, Sobolev and isoperimetric inequalities, and regularity problems of some PDEs.
- When $1 < p < 2$, (G_p) is of different nature. Surprisingly it is always true.

In this talk we focus on the study of (G_p) for $1 < p \leq 2$ and its applications. We start with the setting of graphs. Let (V, E) be an infinite connected graph with symmetric weight μ and let d be the graph distance. Then μ induces a weight on vertices and a measure on the graph. Denote by $p(x, y)$ the transition probability and by P the associated Markov operator. The discrete Laplacian is the operator $I - P$. The (length of) discrete gradient is the "carré du champ" defined as $|\nabla f(x)|^2 = \frac{1}{2} \sum_{y \sim x} p(x, y)|f(x) - f(y)|^2$. Nick Dungey in [6] proved that if (V, E, μ) satisfies the "local doubling" property: $\mu(B(x,1)) \leq C\mu_x$, then for any $1 < p \leq 2$,

$$\||\nabla e^{-t(I-P)} f|\|_p \leq \frac{C}{\sqrt{t}} \|f\|_p. \tag{1}$$

A deep original idea introduced in the proof is the use of "pseudo-gradient":

$$\Gamma_p(f) = pf(I-P)f - f^{2-p}(I-P)f^p,$$

where $1 < p \leq 2$. This notion mimics the chain rule for the Laplace-Beltrami operator on Rieammnian manifolds and is comparable to the carré du champ in certain sense. Hence one can use the analyticity of heat semigroup and Hölder's inequality to deduce the desired gradient estimate.

Motivated by Dungey's proof, we can show that (G_p), $1 < p \leq 2$, always holds on any complete connected Riemannian manifold without any geometry or volume assumption. A significant difference from the discrete case is that the local doubling volume property is not needed. The proof in [2] relies on the chain rule for the Laplace-Beltrami operator on appropriate function u:

$$\Delta u^p(x,t) = -p(p-1)u^{p-2}(x,t)|\nabla u(x,t)|^2 + pu^{p-1}\Delta u,$$

as well as a delicate cut-off argument.

Coming back to the setting of graphs, a natural question to ask is whether or not one can remove the "local doubling" assumption in Dungey's result. Together with T. Coulhon and B. Hua [4], we work on locally finite connected graph (V, E) endowed with a symmetric weight μ on edges and a weight ν on vertices such that $\sup_{x \in V} \frac{\sum_{y \sim x} \mu_{xy}}{\nu_x} < \infty$. The associated bounded Laplacian is defined by

$$\Delta_{\mu,\nu} f = \frac{1}{\nu_x} \sum_{y \sim x} \mu_{xy}(f(x) - f(y)).$$

Considering the gradient on edges $|Df|(\{x,y\}) = |f(y) - f(x)|$, we prove that

$$\||De^{-t\Delta_{\mu,\nu}} f|\|_{\ell^p(E,\mu)} \leq \frac{C}{\sqrt{t}} \|f\|_{\ell^p(V,\nu)}. \tag{2}$$

Our proof adopts a symmetrization argument for the "pseudo-gradient". That is, one writes for $1 < p \leq 2$

$$\Gamma_p(f)(x) = pf\Delta_{\mu,\nu} f - f^{2-p}\Delta_{\mu,\nu}(f^p) = \sum_y \frac{\mu_{xy}}{\nu_x} \gamma_p(f(x), f(y)),$$

where $\gamma_p(\alpha, \beta) = p\alpha(\alpha - \beta) - \alpha^{2-p}(\alpha^p - \beta^p), \forall \alpha, \beta \geq 0$. A crucial observation is that $\gamma_p(\alpha, \beta) + \gamma_p(\beta, \alpha) \simeq (\alpha - \beta)^2$. Hence one can run the gradient estimate by using the analyticity of heat semigroup and Hölder's inequality.

In [1], we further carry (G_p) to Dirichlet spaces. Let X be a good measurable space equipped with a σ-finite measure μ. Let $(\mathcal{E}, \mathcal{F} = \mathrm{Dom}(\mathcal{E}))$ be a Dirichlet form on $L^2(X, \mu)$ and $\{P_t\}_{t>0}$ be the associated heat semigroup. Assume that P_t

is conservative, i.e. $P_t 1 = 1$. Then the analogue of (G_p), $1 < p \le 2$, has the form

$$\|P_t f\|_{p,1/2} \le \frac{C}{\sqrt{t}} \|f\|_p, \tag{3}$$

where $\|\cdot\|_{p,1/2}$ is the seminorm of the heat semigroup-based Besov space introduced in [1]:

$$\mathbf{B}^{p,\alpha}(X) = \left\{ f \in L^p(X), \|f\|_{p,\alpha} := \sup_{t>0} \frac{1}{t^\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} < \infty \right\}.$$

The symmetrization argument in [4] can also be applied in this setting.

The property (G_p), $1 < p \le 2$, on Riemannian manifolds or its substitutes (1), (2) on graphs and (3) on Dirichlet spaces describe the regularity of heat semigroups in different settings. These properties turn to be very powerful tools dealing with problems arising in analysis. We describe two applications here.

- One application is on the L^p boundedness of Riesz transform $\nabla\Delta^{-1/2}$ on Riemannian manifolds or graphs. Assuming volume doubling property and Gaussian heat kernel upper bound, Coulhon and Duong [5] proved that $\nabla\Delta^{-1/2} : L^p \to L^p$ for $1 < p \le 2$. The key ingredient is a weighted version of (G_2) which was proved by Grigor'yan in [7] using integration by parts. Replacing the Gaussian upper bound by a sub-Gaussian one (which is satisfied by some fractal-like manifolds or graphs), we prove in [3] the same results. In this case Grigor'yan's approach does not work anymore. As a natural substitute, we use a weighted version of (G_p) for $1 < p < 2$, which follows from Dungey's idea on the use of chain rule.
- The other application is on the study of critical exponents of Besov spaces, i.e., $\alpha^*(p) = \sup\{\alpha : \mathbf{B}^{p,\alpha} \text{ is nontrivial}\}$. The property (3) leads to a stronger version of pseudo-Poincaré inequality for $p \ge 2$. As a consequence, one can deduce $\alpha^*(p) \le 1/2$ for $p \ge 2$.

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